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Suboptimal Discrete Filters for Stochastic Systems with Different Types of Observations

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Abstract—In [1], we developed a new suboptimal filtering methods for a class of linear and nonlinear continuous dynamic systems with multidimensional observation vector. The methods are based on the decomposition of Kalman filtering and extended Kalman filtering equations by observation vector. In this paper, we present a generalization of these filtering methods to discrete stochastic systems determined by difference equations. The obtained filtering equations have a parallel structure and are very suitable for parallel programming. Example demonstrating the efficiency of the proposed suboptimal filters is given.

Keywords—Discrete system, State vector, Observation vector, Multidimensional vector, Decomposition, Kalman filter, Suboptimal filter.

1. INTRODUCTION

The integration and fusion of information from a combination of different types of measuring instruments (sensors) are often used in the design of high-accuracy control systems. Typical applications that can benefit from the use of multiple sensors are industrial tasks, military command, and control for battlefield management, mobile robot navigation, multitarget tracking, aircraft navigation, and so on. Therefore, the problem of development of multidimensional signal processing algorithms is of great practical importance.

The purpose of this paper is a generalization of suboptimal filtering methods [1] to discrete dynamic systems with different types of observations. The linear Kalman filter and extended Kalman filter are replaced by local filters which are unconnected to each other and allow parallel processing of observations thus reducing off-line and on-line computational requirements. This

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has been achieved via the use of a decomposition of the multidimensional observation vector into a set of subvectors of lower dimension. The numerical example demonstrates the efficiency and high-accuracy of the proposed suboptimal filters.

2. PROBLEM STATEMENT FOR LINEAR SYSTEMS

Consider discrete stochastic system whose state vector $x_k \in R^n$ is determined by linear difference equation

$$x_k = F_{k-1}x_{k-1} + G_{k-1}w_{k-1}, \qquad k = 1, 2, \ldots, \tag{1}$$

where k is discrete time, F_k and G_k are $n \times n$ and $n \times r$ matrices, respectively, $w_k \in R^r$ is a Gaussian random vector, $w_k \sim N(0, Q_k)$, and R^n is an n-dimensional Euclidian space.

Suppose that observation vector $y_k \in R^m$ is composed of N different types of observation subvectors $y_k^{(1)}, \ldots, y_k^{(N)}$,

$$y_k = \left[y_k^{(1)^\top}, \ldots, y_k^{(N)^\top} \right]^\top,$$

where $y_k^{(i)}$, $i = 1, \ldots, N$ are determined by the equations

$$y_k^{(1)} = H_k^{(1)} x_k + v_k^{(1)}, \ldots, y_k^{(N)} = H_k^{(N)} x_k + v_k^{(N)}. \tag{2}$$

Here $y_k^{(i)} \in R^{m_i}$, $H_k^{(i)}$ is $m_i \times n$ matrix, $v_k^{(i)}$ is a Gaussian random vector, $v_k^{(i)} \sim N(0, R_k^{(i)})$, $i = 1, \ldots, N$, $m_1 + \cdots + m_N = m$.

We shall also assume that

(a) $\{w_k\}, \{v_k^{(1)}\}, \ldots, \{v_k^{(N)}\}$ are independent random sequences, i.e., for $i \neq j$, $i, j = 1, \ldots, N$, $E[w_k v_k^{(i)^\top}] = E[v_k^{(i)} v_k^{(j)^\top}] = 0$;

(b) the initial state x_0 is Gaussian random vector, $x_0 \sim N(\overline{x_0}, P_0)$;

(c) x_0 is independent of $\{w_k\}, \{v_k^{(1)}\}, \ldots, \{v_k^{(N)}\}$.

It is required to find mean square estimate of the state vector x_k based on the results of observations

$$Y_k^N = \left\{ y_j^{(i)},\ j = 1, 2, \ldots, k;\ i = 1, \ldots, N \right\}, \qquad \text{for } k = 1, 2, \ldots. \tag{3}$$

3. CONSTRUCTION OF SUBOPTIMAL LINEAR FILTER

Denote the optimal mean square estimate of the state x_k based on observations Y_k^N (3) by $\hat{x}_{k|k}$. The discrete Kalman filtering equations can be used for determining this estimate. For this purpose we rewrite the equations (1) and (2) in the following form:

$$x_k = F_{k-1}x_{k-1} + G_{k-1}w_{k-1}, \tag{4}$$

$$y_k = H_k x_k + v_k, \tag{5}$$

where $y_k \in R^m$, $m = m_1 + \cdots + m_N$, H_k is $m \times n$ matrix, $v_k \in R^m$, $v_k \sim N(0, R_k)$,

$$y_k = \left[y_k^{(1)^\top}, \ldots, y_k^{(N)^\top} \right]^\top, \qquad H_k = \left[H_k^{(1)^\top}, \ldots, H_k^{(N)^\top} \right]^\top,$$

$$v_k = \left[v_k^{(1)^\top}, \ldots, v_k^{(N)^\top} \right]^\top, \quad \text{and} \quad R_k = \operatorname{diag}\left[R_k^{(1)}, \ldots, R_k^{(N)} \right].$$

Applying the discrete Kalman filtering equations to the system (4) and (5), we have the equations determining the optimal mean square estimate $\hat{x}_{k|k}$:

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k \left[y_k - H_k \hat{x}_{k|k-1} \right], \\ \hat{x}_{k|k-1} &= F_{k-1} \hat{x}_{k-1|k-1}, \qquad \hat{x}_{0|0} = \overline{x}_0, \\ P_{k|k-1} &= F_{k-1} P_{k-1|k-1} F_{k-1}^{\top} + G_{k-1} Q_{k-1} G_{k-1}^{\top}, \\ K_k &= P_{k|k-1} H_k^{\top} \left[H_k P_{k|k-1} H_k^{\top} + R_k \right]^{-1}, \\ P_{k|k} &= \left[I - K_k H_k \right] P_{k|k-1}, \qquad P_{1|0} = P_0, \end{aligned} \tag{6}$$

where I is unit matrix, $P_{k|k}$ is the covariance matrix of the filtering error $\tilde{x}_{k|k}$,

$$P_{k|k} = E\left[\tilde{x}_{k|k}\, \tilde{x}_{k|k}^{\top} \right] \quad \text{and} \quad \tilde{x}_{k|k} = \hat{x}_{k|k} - x_k. \tag{7}$$

The Equations for a Suboptimal Linear Filter

The derivation of new suboptimal filtering equations is based on the assumption that the observation vector y_k consists of the combination of different types of observations $y_k^{(1)}, \dots, y_k^{(N)}$ [1]. According to (1) and (2), we have N unconnected subsystems ($i = 1, \dots, N$) with state vector $x_k \in R^n$ and observation vector $y_k^{(i)} \in R^{m_i}$:

$$x_k = F_{k-1} x_{k-1} + G_{k-1} w_{k-1}, \tag{8}$$

$$y_k^{(i)} = H_k^{(i)} x_k + v_k^{(i)}, \tag{9}$$

where i (the number of subsystems) is fixed.

We denote the estimate of state vector x_k based on observations $\{y_1^{(i)}, \dots, y_k^{(i)}\}$ by $\hat{x}_{k|k}^{(i)}$. To find the estimate $\hat{x}_{k|k}^{(i)}$ we apply the Kalman filtering equations to linear subsystem (8) and (9). We have

$$\begin{aligned} \hat{x}_{k|k}^{(i)} &= \hat{x}_{k|k-1}^{(i)} + K_k^{(i)} \left[y_k^{(i)} - H_k^{(i)} \hat{x}_{k|k-1}^{(i)} \right], \\ \hat{x}_{k|k-1}^{(i)} &= F_{k-1} \hat{x}_{k-1|k-1}^{(i)}, \qquad \hat{x}_{0|0}^{(i)} = \overline{x}_0, \\ P_{k|k-1}^{(i)} &= F_{k-1} P_{k-1|k-1}^{(i)} F_{k-1}^{\top} + G_{k-1} Q_{k-1} G_{k-1}^{\top}, \\ K_k^{(i)} &= P_{k|k-1}^{(i)} H_k^{(i)\top} \left[H_k^{(i)} P_{k|k-1}^{(i)} H_k^{(i)\top} + R_k^{(i)} \right]^{-1}, \\ P_{k|k}^{(i)} &= \left[I - K_k^{(i)} H_k^{(i)} \right] P_{k|k-1}^{(i)}, \qquad P_{1|0}^{(i)} = P_0. \end{aligned} \tag{10}$$

In equation (10), $P_{k|k}^{(i)}$ is the covariance matrix of filtering error $\Delta x_{k|k}^{(i)}$,

$$\begin{aligned} P_{k|k}^{(i)} &= E\left[\Delta x_{k|k}^{(i)}\, \Delta x_{k|k}^{(i)\top} \right], \\ \Delta x_{k|k}^{(i)} &= \hat{x}_{k|k}^{(i)} - x_k. \end{aligned} \tag{11}$$

Note that the estimate $\hat{x}_{k|k}^{(i)}$ is unbiased, i.e.,

$$E\left[\hat{x}_{k|k}^{(i)} \right] = E x_k, \qquad \text{for any } i = 1, \dots, N; \quad k = 0, 1, \dots. \tag{12}$$

Thus, from equations (10) we have N filtering estimates $\hat{x}_{k|k}^{(1)}, \dots, \hat{x}_{k|k}^{(N)}$ based on the observations $\{y_1^{(1)}, \dots, y_k^{(1)}\}, \dots, \{y_1^{(N)}, \dots, y_k^{(N)}\}$, respectively. Then the new suboptimal estimate $\hat{x}_{k|k}^{*}$

of the state vector x_k based on the all observations $Y_k^N(3)$ is constructed from the estimates $\hat{x}_{k|k}^{(1)}, \dots, \hat{x}_{k|k}^{(N)}$ by the following formula:

$$\hat{x}_{k|k}^{*} = \sum_{i=1}^{N} c_k^{(i)} \hat{x}_{k|k}^{(i)}, \qquad \sum_{i=1}^{N} c_k^{(i)} = I, \tag{13}$$

where $c_k^{(1)}, \dots, c_k^{(N)}$ are weighting coefficients (matrices) determined by the mean square criterion,

$$\left| \sum_{i=1}^{N} c_k^{(i)} \hat{x}_{k|k}^{(i)} - x_k \right|^2 \to \min_{c_k^{(i)}}.$$

The equations for optimal coefficients $c_k^{(i)}$, $i = 1, \dots, N$, have the form

$$\begin{gathered} \sum_{i=1}^{N-1} c_k^{(i)} \left(P_{k|k}^{(ij)} - P_{k|k}^{(iN)} \right) + c_k^{(N)} \left(P_{k|k}^{(Nj)} - P_{k|k}^{(NN)} \right) = 0, \\ \text{for } j = 1, \dots, N-1, \qquad c_k^{(1)} + \dots + c_k^{(N)} = I, \end{gathered} \tag{14}$$

where $P_{k|k}^{(ij)}$ is cross-covariance matrix of the filtering errors $\Delta x_{k|k}^{(i)}$ and $\Delta x_{k|k}^{(j)}$ at $i \neq j$ and $P_{k|k}^{(ii)}$ is covariance matrix of the filtering error $\Delta x_{k|k}^{(i)}$, i.e.,

$$\begin{aligned} P_{k|k}^{(ij)} &= E\left[\Delta x_{k|k}^{(i)} \, \Delta x_{k|k}^{(j)\,\mathsf{T}} \right], \\ P_{k|k}^{(ii)} &= P_{k|k}^{(i)} = E\left[\Delta x_{k|k}^{(i)} \, \Delta x_{k|k}^{(i)\,\mathsf{T}} \right]. \end{aligned} \tag{15}$$

The set of equations (14) represents the linear algebraic equations for the matrices $c_k^{(1)}, \dots, c_k^{(N)}$ for any $k = 1, 2, \dots$. In the particular case of uncorrelated errors ($P_{k|k}^{(ij)} = 0, i \neq j$) the equations (14) coincide with the analogous equations for $c_k^{(i)}, i = 1, \dots, N$ given in [2,3] (for $N = 2$) and in [1] (for arbitrary N). The covariance matrix $P_{k|k}^{(ii)} = P_{k|k}^{(i)}$ in (14) is determined by Kalman filtering equations (10). The equations for the cross-covariance matrices $P_{k|k}^{(ij)} (i \neq j)$ in (14) take the form:

$$\begin{aligned} P_{k|k}^{(ij)} &= \left[I - K_k^{(i)} H_k^{(i)} \right] \left(F_{k-1} P_{k-1|k-1}^{(ij)} F_{k-1}^{\mathsf{T}} + G_{k-1} Q_{k-1} G_{k-1}^{\mathsf{T}} \right) \left[I - K_k^{(j)} H_k^{(j)} \right]^{\mathsf{T}}, \\ P_{0|0}^{(ij)} &= P_0, \quad \text{for all } i \neq j, \ i, j = 1, \dots, N, \end{aligned} \tag{16}$$

where the matrix (gain) $K_k^{(i)}$ is determined by equations (10). The derivation of equations (16) is given in the Appendix.

Thus the equations (10), (13), (14), and (16) completely define the new suboptimal linear filter for estimate $\hat{x}_{k|k}^{*}$ of the state vector x_k. Note that the equations (10) are separated for various values of $i = 1, \dots, N$. Therefore, it can be solved in parallel.

REMARKS.

(a) The estimate $\hat{x}_{k|k}^{*}$ (13) is unbiased. Using (13), we obtain

$$\begin{aligned} E\hat{x}_{k|k}^{*} &= E\left\{ \sum_{i=1}^{N} c_k^{(i)} E\left[\hat{x}_{k|k}^{(i)} \right] \right\} \\ &= \left[\sum_{i=1}^{N} c_k^{(i)} \right] E x_k \\ &= E x_k, \qquad \text{for } k = 0, 1, \dots. \end{aligned}$$

(b) The suboptimal filtering equations (10), (13), (14), and (16) require block diagonal covariance matrix $R_k = \text{diag}[R_k^{(1)}, \dots, R_k^{(N)}]$ of composite measurement noise $v_k = [v_k^{(1)^\top}, \dots, v_k^{(N)^\top}]^\top$. If R_k is not block diagonal (in the case of correlated noises $v_k^{(1)}, \dots, v_k^{(N)}$), the diagonalization methods can be used to find a linear transformation of the observation vector $y_k = [y_k^{(1)^\top} \dots y_k^{(N)^\top}]^\top$, whose measurements noise will be block diagonal [4].

The Accuracy of Suboptimal Linear Filter

Now we derive the equation for the actual covariance matrix

$$P_{k|k}^* = E\left[\Delta x_{k|k} \Delta x_{k|k}^\top\right], \qquad \Delta x_{k|k} = \hat{x}_{k|k}^* - x_k, \tag{17}$$

where x_k is the state vector (1) and $\hat{x}_{k|k}^*$ is the suboptimal filtering estimate (13).

Substituting (13) into (17) and taking into account that $c_k^{(1)} + \cdots + c_k^{(N)} = I$ we have

$$P_{k|k}^* = \sum_{i,j=1}^{N} c_k^{(i)} P_{k|k}^{(ij)} c_k^{(j)^\top}, \tag{18}$$

where the covariance matrices $P_{k|k}^{(ij)}$ are determined by equations (10) (at $i = j$) and equations (16) (at $i \neq j$), respectively.

Thus, the actual covariance matrix of the filtering error $P_{k|k}^*$ and the actual mean square error $\text{tr}[P_{k|k}^*]$ can be calculated by using the formula (18) and equations (10) and (16).

4. EXAMPLE OF LINEAR SYSTEM

Consider the problem of recursive estimation of an unknown scalar parameter [3,5]. To estimate the value of the unknown parameter x from two types of observations corrupted by additive white noises, the system, and observation models are given by

$$x_k = x_{k-1}, \tag{19}$$

$$y_k^{(1)} = x_k + v_k^{(1)}, \qquad y_k^{(2)} = x_k + v_k^{(2)}, \tag{20}$$

where x_k, $y_k^{(1)}$, $y_k^{(2)} \in R$, and $v_k^{(1)} \sim N(0, r_1)$, $v_k^{(2)} \sim N(0, r_2)$ are independent Gaussian noises. Let $x_0 \sim N(\overline{x}_0, P_0)$.

The optimal filtering estimate $\hat{x}_{k|k}$ based on observations (20)

$$Y_k^{(1)} = \left\{y_1^{(1)}, \dots, y_k^{(1)}\right\}, \qquad Y_k^{(2)} = \left\{y_1^{(2)}, \dots, y_k^{(2)}\right\}$$

is determined by the discrete Kalman filter. Substituting in (6)

$$F_k = 1, \quad G_k = Q_k = 0, \quad H_k = [1 \ \ 1]^\top, \quad R_k = \text{diag}[r_1, \ r_2], \quad K_k = [K_{1,k} \ \ K_{2,k}]$$

we have

$$\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k-1|k-1} + K_{1,k}\left(y_k^{(1)} - \hat{x}_{k-1|k-1}\right) + K_{2,k}\left(y_k^{(2)} - \hat{x}_{k-1|k-1}\right), \\
\hat{x}_{0|0} &= \overline{x}_0, \\
P_{k|k-1} &= P_{k-1|k-1}, \qquad P_{0|0} = P_0, \\
K_{1,k} &= \frac{r_2 P_{k|k-1}}{r_1 r_2 + (r_1 + r_2) P_{k|k-1}}, \\
K_{2,k} &= \frac{r_1 P_{k|k-1}}{r_1 r_2 + (r_1 + r_2) P_{k|k-1}}, \\
P_{k|k} &= \frac{r_1 r_2 P_{k|k-1}}{r_1 r_2 + (r_1 + r_2) P_{k|k-1}}.
\end{aligned} \tag{21}$$

Using the "step by step" approach [5], we obtain the exact formula for the actual variance of the error $P_{k|k}$,

$$\begin{aligned} P_{k|k} &= E\left[\hat{x}_{k|k} - x_k\right]^2 = \frac{P_0}{1 + k r_{12} P_0}, \\ r_{12} &= \frac{r_1 + r_2}{r_1 r_2}. \end{aligned} \tag{22}$$

Together with the optimal Kalman filter (21), we apply the suboptimal filter (10), (13), (14), and (16). We denote the estimates of the unknown parameter x_k based on observations Y_k^1 and Y_k^2 by $\hat{x}_{k|k}^{(1)}$ and $\hat{x}_{k|k}^{(2)}$, respectively. Using the equations (10) for $i = 1, 2$, we obtain the equations for $\hat{x}_{k|k}^{(1)}$ and $\hat{x}_{k|k}^{(2)}$,

$$\begin{aligned} \hat{x}_{k|k}^{(1)} &= \hat{x}_{k-1|k-1}^{(1)} + K_k^{(1)}\left(y_k^{(1)} - \hat{x}_{k-1|k-1}^{(1)}\right), \qquad \hat{x}_{0|0}^{(1)} = \overline{x}_0, \\ P_{k|k-1}^{(1)} &= P_{k-1|k-1}^{(1)}, \qquad P_{0|0}^{(1)} = P_0, \\ K_k^{(1)} &= \frac{P_{k|k-1}^{(1)}}{P_{k|k-1}^{(1)} + r_1}, \\ P_{k|k}^{(1)} &= \left[1 - K_k^{(1)}\right] P_{k|k-1}^{(1)} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \hat{x}_{k|k}^{(2)} &= \hat{x}_{k-1|k-1}^{(2)} + K_k^{(2)}\left(y_k^{(2)} - \hat{x}_{k-1|k-1}^{(2)}\right), \qquad \hat{x}_{0|0}^{(2)} = \overline{x}_0, \\ P_{k|k-1}^{(2)} &= P_{k-1|k-1}^{(2)}, \qquad P_{0|0}^{(2)} = P_0, \\ K_k^{(2)} &= \frac{P_{k|k-1}^{(2)}}{P_{k|k-1}^{(2)} + r_2}, \\ P_{k|k}^{(2)} &= \left[1 - K_k^{(2)}\right] P_{k|k-1}^{(2)}. \end{aligned} \tag{24}$$

The exact formulae for $P_{k|k}^{(1)}$ and $P_{k|k}^{(2)}$ take the form [5]:

$$\begin{aligned} P_{k|k}^{(1)} &= E\left[\hat{x}_{k|k}^{(1)} - x_k\right]^2 = \frac{r_1 P_0}{r_1 + k r_1 P_0}, \\ P_{k|k}^{(2)} &= E\left[\hat{x}_{k|k}^{(2)} - x_k\right]^2 = \frac{r_2 P_0}{r_2 + k r_2 P_0}. \end{aligned} \tag{25}$$

Then from equations (13) and (14) at $N = 2$, the suboptimal estimate $\hat{x}_{k|k}^*$ takes the form

$$\hat{x}_{k|k}^* = c_k^{(1)} \hat{x}_{k|k}^{(1)} + c_k^{(2)} \hat{x}_{k|k}^{(2)}, \tag{26}$$

$$\begin{aligned} c_k^{(1)} &= \frac{P_{k|k}^{(2)} - P_{k|k}^{(12)}}{P_{k|k}^{(1)} - 2P_{k|k}^{(12)} + P_{k|k}^{(2)}}, \\ c_k^{(2)} &= \frac{P_{k|k}^{(1)} - P_{k|k}^{(12)}}{P_{k|k}^{(1)} - 2P_{k|k}^{(12)} + P_{k|k}^{(2)}}, \end{aligned} \tag{27}$$

where $P_{k|k}^{(1)}$ and $P_{k|k}^{(2)}$ are determined by the formulae (25). According to (16) the correlation $P_{k|k}^{(12)}$ is determined by equation

$$P_{k|k}^{(12)} = \left(1 - K_k^{(1)}\right)\left(1 - K_k^{(2)}\right) P_{k-1|k-1}^{(12)}, \qquad P_{0|0}^{(12)} = P_0. \tag{28}$$

Table 1.

k	$P_{k\|k}^{(1)}$	$P_{k\|k}^{(12)}$	$P_{k\|k}^{(2)}$	$c_k^{(1)}$	$c_k^{(2)}$	$P_{k\|k}^{*}$	$P_{k\|k}$	$\epsilon_k(\%)$
0	2	2	2	–	–	2	2	0
1	0.6666	0.3333	1	$\frac{2}{3}$	$\frac{1}{3}$	0.5555	0.5	11.1
2	0.4	0.1333	0.6666	$\frac{2}{3}$	$\frac{1}{3}$	0.3111	0.2857	8.8
3	0.2857	0.0714	0.5	$\frac{2}{3}$	$\frac{1}{3}$	0.2142	0.2	7.1
4	0.2222	0.0444	0.4	$\frac{2}{3}$	$\frac{1}{3}$	0.1629	0.1538	5.9
5	0.1818	0.0303	0.3333	$\frac{2}{3}$	$\frac{1}{3}$	0.1313	0.125	5.0
6	0.1538	0.0219	0.2857	$\frac{2}{3}$	$\frac{1}{3}$	0.1098	0.1053	4.4
7	0.1333	0.0166	0.25	$\frac{2}{3}$	$\frac{1}{3}$	0.0944	0.0909	3.9
8	0.1176	0.0130	0.2222	$\frac{2}{3}$	$\frac{1}{3}$	0.0827	0.08	3.4
9	0.1052	0.0105	0.2	$\frac{2}{3}$	$\frac{1}{3}$	0.0736	0.0714	3.1
10	0.0952	0.0086	0.1818	$\frac{2}{3}$	$\frac{1}{3}$	0.0663	0.0645	2.8

According to (18) the actual variance of error $P_{k|k}^{*} = E[x_{k|k}^{*} - x_k]^2$ (17) of the suboptimal estimate (26) has the form:

$$P_{k|k}^{*} = \left[c_k^{(1)}\right]^2 P_{k|k}^{(1)} + 2c_k^{(1)}c_k^{(2)}P_{k|k}^{(12)} + \left[c_k^{(2)}\right]^2 P_{k|k}^{(2)}. \tag{29}$$

The formulae (25), (27), (29), and the equation (28) produce the actual accuracy of the suboptimal filter (23), (24), (26), and (27). The results of calculations were performed at the values of parameters: $r_1 = 1$, $r_2 = 2$, $P_0 = 2$.

$\epsilon_k = \mid (P_{k|k} - P_{k|k}^{*})/P_{k|k} \mid 100\%$ is relative error of the suboptimal filter.

As is seen from Table 1, the variance of error $P_{k|k}^{*}$ (29) of the suboptimal filter is practically the same as the variance of error $P_{k|k}$ (22) of the Kalman filter (21).

5. SUBOPTIMAL FILTER FOR NONLINEAR SYSTEMS

The linear suboptimal filter (10), (13), (14), and (16) is generalized to nonlinear discrete systems determined by equations

$$x_k = f_{k-1}(x_{k-1}) + g_{k-1}(x_{k-1})w_{k-1}, \qquad k = 1, 2, \ldots, \tag{30}$$

$$y_k^{(1)} = h_k^{(1)}(x_k) + v_k^{(1)}, \ldots, y_k^{(N)} = h_k^{(N)}(x_k) + v_k^{(N)}, \tag{31}$$

where $x_k \in R^n$, $w_k \in R^r \sim N(0, Q_k)$, $y_k \in R^{m_k}$, $v_k \in R^{m_k} \sim N(0, R_k)$, $k = 1, \ldots, N$, and f_k, g_k, and h_k are given vector (matrix) functions.

According to the main idea of linear suboptimal filtering (Section 3) we apply the extended Kalman filtering equations to the nonlinear subsystem with state vector x_k and observation $y_k^{(i)}$ (i is fixed):

$$\begin{aligned} x_k &= f_{k-1}(x_{k-1}) + g_{k-1}(x_{k-1})w_{k-1}, \\ y_k^{(i)} &= h_k^{(i)}(x_k) + v_k^{(i)}. \end{aligned} \tag{32}$$

We have

$$
\begin{aligned}
\hat{x}_{k|k}^{(i)} &= \hat{x}_{k|k-1}^{(i)} + K_k^{(i)} \left[y_k^{(i)} - h_k^{(i)} \left(\hat{x}_{k|k-1}^{(i)} \right) \right], \\
\hat{x}_{k|k-1}^{(i)} &= f_{k-1} \left(\hat{x}_{k-1|k-1}^{(i)} \right), \qquad \hat{x}_{0|0}^{(i)} = \overline{x}_0, \\
P_{k|k-1}^{(i)} &= F_{k-1}^{(i)} P_{k-1|k-1}^{(i)} F_{k-1}^{(i)\,\top} + g_{k-1} \left(\hat{x}_{k-1|k-1}^{(i)} \right) Q_{k-1} \left(g_{k-1} \left(\hat{x}_{k-1|k-1}^{(i)} \right) \right)^\top, \\
K_k^{(i)} &= P_{k|k-1}^{(i)} H_k^{(i)\top} \left[H_k^{(i)} P_{k|k-1}^{(i)} H_k^{(i)\top} + R_k^{(i)} \right]^{-1}, \\
P_{k|k}^{(i)} &= \left[I - K_k^{(i)} H_k^{(i)} \right] P_{k|k-1}^{(i)}, \qquad P_{1|0}^{(i)} = P_0,
\end{aligned}
\tag{33}
$$

where

$$
H_k^{(i)} = \left. \frac{\partial h_k}{\partial x} \right|_{x=\hat{x}_{k|k-1}^{(i)}}, \qquad F_{k-1}^{(i)} = \left. \frac{\partial f_{k-1}}{\partial x} \right|_{x=\hat{x}_{k-1|k-1}^{(i)}}. \tag{34}
$$

From equation (33), we have N estimates $\hat{x}_{k|k}^{(1)}, \dots, \hat{x}_{k|k}^{(N)}$ based on observations $\{y_1^{(1)}, \dots, y_k^{(1)}\}$, $\dots, \{y_1^{(N)}, \dots, y_k^{(N)}\}$, respectively. And finally the suboptimal estimate $\hat{x}_{k|k}^*$ of the state vector x_k based on the all observations Y_k^N (3) is determined by the formula (13):

$$
\hat{x}_{k|k}^* = \sum_{i=1}^{N} c_k^{(i)} \hat{x}_{k|k}^{(i)},
$$

where the weighting coefficients $c_k^{(1)}, \dots, c_k^{(N)}$ satisfy the equations (14).

Thus the equations (33), (34), (14), and (16) completely define the new suboptimal filter for the nonlinear system (32).

6. EXAMPLE FOR NONLINEAR SYSTEM

The discrete approximation of equations of motion [3] with two independent measurement sensors has the form:

$$
\begin{aligned}
x_k &= x_{k-1} - \Delta t \sin(x_{k-1}) + w_{k-1}, \\
y_k^{(1)} &= \frac{1}{2} \sin(2x_k) + v_k^{(1)}, \\
y_k^{(2)} &= \frac{1}{2} \sin(2x_k) + v_k^{(2)},
\end{aligned}
\tag{35}
$$

where Δt is the step of approximation, x_k, $y_k^{(1)}$, $y_k^{(2)} \in R$, and $w_k \sim N(0, q)$, $v_k^{(1)} \sim N(0, r_1)$, $v_k^{(2)} \sim N(0, r_2)$.

For finding the estimate $\hat{x}_{k|k}$ of the state vector x_k based on the all observations $Y_k^1 = \{y_1^{(1)}, \dots, y_k^{(1)}\}$ and $Y_k^2 = \{y_1^{(2)}, \dots, y_k^{(2)}\}$ we shall apply the extended Kalman filter (33),(34) to the system (35). Taking into account that

$$
\begin{aligned}
f_{k-1}(x_{k-1}) &= x_{k-1} - \Delta t \sin(x_{k-1}), \\
g_{k-1}(x_{k-1}) &= 1, \qquad Q_k = q, \\
h_k(x_k) &= [h_{1,k}(x_k) h_{2,k}(x_k)]^\top, \\
y_k &= \left[y_k^{(1)} \quad y_k^{(2)} \right]^\top, \\
h_{1,k}(x_k) &= h_{2,k}(x_k) = \frac{1}{2} \sin(2x_k), \\
R_k &= \operatorname{diag}(r_1, r_2),
\end{aligned}
$$

$$F_{k-1}(x) = \frac{\partial f_{k-1}}{\partial x} = 1 - \Delta t \cos(x), H_k(x) = \left[\frac{\partial h_{1,k}}{\partial x} \quad \frac{\partial h_{2,k}}{\partial x}\right]^{\mathsf{T}},$$
$$\frac{\partial h_{1,k}}{\partial x} = \frac{\partial h_{2,k}}{\partial x} = \cos(2x)$$

we have

$$\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{1,k}\left[y_k^{(1)} - h_{1,k}(\hat{x}_{k|k-1})\right] + K_{2,k}\left[y_k^{(2)} - h_{2,k}(\hat{x}_{k|k-1})\right], \\
K_k &= [K_{1,k} \quad K_{2,k}], \\
\hat{x}_{k|k-1} &= f_{k-1}\left(\hat{x}_{k-1|k-1}\right), \\
P_{k|k-1} &= F_{k-1}^2 P_{k-1|k-1} + q, \\
K_k &= P_{k|k-1} H_k^{\mathsf{T}}\left[H_k P_{k|k-1} H_k^{\mathsf{T}} + R_k\right]^{-1}, \\
P_{k|k} &= [1 - K_k H_k] P_{k|k-1},
\end{aligned} \tag{36}$$

where the derivatives F_{k-1} and H_k are calculated at the points $\hat{x}_{k-1|k-1}$ and $\hat{x}_{k|k-1}$, respectively, i.e., $F_{k-1} = F_{k-1}(\hat{x}_{k-1|k-1})$, $H_k = H_k(\hat{x}_{k|k-1})$.

Together with the extended Kalman filter (36) let us apply the suboptimal filter (33) and (34) based on the decomposition of observation vector $y_k = [y_k^{(1)} \ y_k^{(2)}]^{\mathsf{T}}$. In this case, the suboptimal estimate $\hat{x}^*_{k|k}$ of the state variable x_k is determined by

$$\hat{x}^*_{k|k} = c_k^{(1)} \hat{x}_{k|k}^{(1)} + c_k^{(2)} \hat{x}_{k|k}^{(2)}, \tag{37}$$

where $\hat{x}_{k|k}^{(1)}$ and $\hat{x}_{k|k}^{(2)}$ are the estimates of x_k based on observations Y_k^1 and Y_k^2, respectively. From (33) for $i = 1$ we have

$$\begin{aligned}
\hat{x}_{k|k}^{(1)} &= \hat{x}_{k|k-1}^{(1)} + K_k^{(1)}\left[y_k^{(1)} - h_k^{(1)}\left(\hat{x}_{k|k-1}^{(1)}\right)\right], \\
\hat{x}_{k|k-1}^{(1)} &= f_{k-1}\left(\hat{x}_{k-1|k-1}^{(1)}\right), \\
P_{k|k-1}^{(1)} &= {F_{k-1}^{(1)}}^2 P_{k-1|k-1}^{(1)} + q, \\
K_k^{(1)} &= \frac{P_{k|k-1}^{(1)} H_k^{(1)}}{{H_k^{(1)}}^2 P_{k|k-1}^{(1)} + r_1}, \\
P_{k|k}^{(1)} &= \left(1 - K_k^{(1)} H_k^{(1)}\right) P_{k|k-1}^{(1)},
\end{aligned} \tag{38}$$

and analogously for $i = 2$,

$$\begin{aligned}
\hat{x}_{k|k}^{(2)} &= \hat{x}_{k|k-1}^{(2)} + K_k^{(2)}\left[y_k^{(2)} - h_k^{(2)}\left(\hat{x}_{k|k-1}^{(2)}\right)\right], \\
\hat{x}_{k|k-1}^{(2)} &= f_{k-1}\left(\hat{x}_{k-1|k-1}^{(2)}\right), \\
P_{k|k-1}^{(2)} &= {F_{k-1}^{(2)}}^2 P_{k-1|k-1}^{(2)} + q, \\
K_k^{(2)} &= \frac{P_{k|k-1}^{(2)} H_k^{(2)}}{{H_k^{(2)}}^2 P_{k|k-1}^{(2)} + r_2}, \\
P_{k|k}^{(2)} &= \left(1 - K_k^{(2)} H_k^{(2)}\right) P_{k|k-1}^{(2)}.
\end{aligned} \tag{39}$$

In equations (38) and (39)

$$
\begin{aligned}
f_{k-1}\left(\hat{x}^{(1)}_{k-1|k-1}\right) &= \hat{x}^{(1)}_{k-1|k-1} - \Delta t \sin\left(\hat{x}^{(1)}_{k-1|k-1}\right), \\
f_{k-1}\left(\hat{x}^{(2)}_{k-1|k-1}\right) &= \hat{x}^{(2)}_{k-1|k-1} - \Delta t \sin\left(\hat{x}^{(2)}_{k-1|k-1}\right), \\
h^{(1)}_k\left(\hat{x}^{(1)}_{k|k-1}\right) &= \frac{1}{2}\sin\left(2\hat{x}^{(1)}_{k|k-1}\right), \\
h^{(2)}_k\left(\hat{x}^{(2)}_{k|k-1}\right) &= \frac{1}{2}\sin\left(2\hat{x}^{(2)}_{k|k-1}\right), \\
F^{(1)}_{k-1} &= 1 - \Delta t \cos\left(\hat{x}^{(1)}_{k-1|k-1}\right), \\
F^{(2)}_{k-1} &= 1 - \Delta t \cos\left(\hat{x}^{(2)}_{k-1|k-1}\right), \\
H^{(1)}_k &= \cos\left(2\hat{x}^{(1)}_{k|k-1}\right), \\
H^{(2)}_k &= \cos\left(2\hat{x}^{(2)}_{k|k-1}\right).
\end{aligned}
\tag{40}
$$

Thus, the formula (37) and the equations (38)–(40) completely define the suboptimal filtering estimate $\hat{x}^*_{k|k}$ of the state x_k based on all observations Y^1_k and Y^2_k.

7. CONCLUSION

The new suboptimal filtering equations were derived for a wide class of linear and nonlinear discrete systems determined by difference equations. The equations have a parallel structure. Therefore parallel computers can be used in the design of these filters.

The suboptimal filters with different types of observations can be widely used in the different areas of applications: industrial, military, space, target tracking, inertial navigation, and others [6].

APPENDIX

Derivation of Equations (16)

From equations (8), (9), (10), and (11) we have a linear difference equation for the filtering error $\Delta x^{(i)}_{k|k}$:

$$
\begin{aligned}
\Delta x^{(i)}_{k|k} &= \hat{x}^{(i)}_{k|k} - x_k \\
&= F_{k-1}\hat{x}^{(i)}_{k-1|k-1} + K^{(i)}_k\left[H^{(i)}_k x_k + v^{(i)}_k - H^{(i)}_k F_{k-1}\hat{x}^{(i)}_{k-1|k-1}\right] \\
&\quad - F_{k-1}x_{k-1} - G_{k-1}w_{k-1} \\
&= F_{k-1}\hat{x}^{(i)}_{k-1|k-1} + K^{(i)}_k\left[H^{(i)}_k (F_{k-1}x_{k-1} + G_{k-1}w_{k-1}) + v^{(i)}_k - H^{(i)}_k F_{k-1}\hat{x}^{(i)}_{k-1|k-1}\right] \\
&\quad - F_{k-1}x_{k-1} - G_{k-1}w_{k-1} \\
&= F_{k-1}\Delta x^{(i)}_{k-1|k-1} + K^{(i)}_k\left[-H^{(i)}_k F_{k-1}\Delta x^{(i)}_{k-1|k-1} + H^{(i)}_k G_{k-1}w_{k-1} + v^{(i)}_k\right] \\
&\quad - G_{k-1}w_{k-1} \\
&= \left[I - K^{(i)}_k H^{(i)}_k\right] F_{k-1}\Delta x^{(i)}_{k-1|k-1} + K^{(i)}_k H^{(i)}_k G_{k-1}w_{k-1} \\
&\quad + K^{(i)}_k v^{(i)}_k - G_{k-1}w_{k-1} \\
&= \left[I - K^{(i)}_k H^{(i)}_k\right] F_{k-1}\Delta x^{(i)}_{k-1|k-1} - \left[I - K^{(i)}_k H^{(i)}_k\right] G_{k-1}w_{k-1} + K^{(i)}_k v^{(i)}_k.
\end{aligned}
\tag{A.1}
$$

Hence,

$$
\begin{aligned}
P_{k|k}^{(ij)} &= E\left[\Delta x_{k|k}^{(i)}\, \Delta x_{k|k}^{(j)\,\top}\right] \\
&= E\left\{\left[I - K_k^{(i)} H_k^{(i)}\right] F_{k-1} \Delta x_{k-1|k-1}^{(i)} - \left[I - K_k^{(i)} H_k^{(i)}\right] G_{k-1} w_{k-1} + K_k^{(i)} v_k^{(i)}\right\} \\
&\quad \times \left\{\left[I - K_k^{(j)} H_k^{(j)}\right] F_{k-1} \Delta x_{k-1|k-1}^{(j)} - \left[I - K_k^{(j)} H_k^{(j)}\right] G_{k-1} w_{k-1} + K_k^{(j)} v_k^{(j)}\right\}^{\top}.
\end{aligned} \tag{A.2}
$$

Taking into account that $\Delta x_{k-1|k-1}^{(i)}, w_{k-1}, v_k^{(i)}$, and $v_k^{(j)} (i \neq j)$ are independent random vectors, we have linear difference equation for $P_{k|k}^{(ij)}$:

$$
\begin{aligned}
P_{k|k}^{(ij)} &= \left[I - K_k^{(i)} H_k^{(i)}\right] \left(F_{k-1} P_{k-1|k-1}^{(ij)} F_{k-1}^{\top} + G_{k-1} Q_{k-1} G_{k-1}^{\top}\right) \left[I - K_k^{(j)} H_k^{(j)}\right]^{\top}, \\
P_{0|0}^{(ij)} &= P_0, \qquad i \neq j, \qquad i, j = 1, \dots, N.
\end{aligned} \tag{A.3}
$$

This proves equation (16).

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